

Frustrated superconductivity and intrinsic reduction of T_c in trilayer nickelate

Qiong Qin,^{1,2,5} Jiangfan Wang,^{3,5} and Yi-feng Yang^{1,2,4,*}

*Correspondence: yifeng@iphy.ac.cn

Received: July 7, 2024; Accepted: October 29, 2024; Published Online: November 7, 2024; <https://doi.org/10.59717/j.xinn-mater.2024.100102>

© 2024 The Author(s). This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

GRAPHICAL ABSTRACT



PUBLIC SUMMARY

- The trilayer nickelate exhibits frustrated superconductivity with competing interlayer pairings.
- The superconductivity has a complex doublet order parameter like the Higgs field in particle physics.
- The frustration reduces its superconducting T_c by over half compared to that of bilayer nickelates.
- The interlayer pairing causes a Josephson frequency half that of usual intralayer pairing superconductors.

Frustrated superconductivity and intrinsic reduction of T_c in trilayer nickelate

Qiong Qin,^{1,2,5} Jiangfan Wang,^{3,5} and Yi-feng Yang^{1,2,4,*}

¹Beijing National Laboratory for Condensed Matter Physics and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China

²University of Chinese Academy of Sciences, Beijing 100049, China

³School of Physics, Hangzhou Normal University, Hangzhou 311121, China

⁴Songshan Lake Materials Laboratory, Dongguan 523808, China

⁵These authors contributed equally to this work

*Correspondence: yifeng@iphy.ac.cn

Received: July 7, 2024; Accepted: October 29, 2024; Published Online: November 7, 2024; <https://doi.org/10.59717/j.xinn-mater.2024.100102>

© 2024 The Author(s). This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

Citation: Qin Q., Wang J., and Yang Y (2024). Frustrated superconductivity and intrinsic reduction of T_c in trilayer nickelate. *The Innovation Materials* 2(4): 100102.

Identifying the key factors controlling the magnitude of T_c is of critical importance in the pursuit of high-temperature superconductivity. In cuprates, T_c reaches its maximal value in trilayer structure, leading to the belief that interlayer coupling may help promote the pairing. In contrast, for the recently discovered nickelate superconductors under high pressure, the maximum T_c is reduced from about 80 K in the bilayer $\text{La}_3\text{Ni}_2\text{O}_7$ to 30 K in the trilayer $\text{La}_4\text{Ni}_3\text{O}_{10}$. Motivated by this opposite trend, we propose an interlayer pairing scenario for the superconductivity of $\text{La}_4\text{Ni}_3\text{O}_{10}$. Our theory reveals intrinsic frustration in the spin-singlet pairing that the inner layer tends to form with both of the two outer layers respectively, leading to strong superconducting fluctuations between layers. This explains the reduction of its maximum T_c compared to that of the bilayer $\text{La}_3\text{Ni}_2\text{O}_7$. Our findings support a fundamental distinction between multilayer nickelate and cuprate superconductors, and ascribe it to their different (interlayer versus intralayer) pairing mechanisms. Furthermore, our theory predicts extended s -wave gap structures in $\text{La}_4\text{Ni}_3\text{O}_{10}$, with varying signs and possible nodes on different Fermi pockets. We also find an intrinsic Josephson coupling with potentially interesting consequences that may be examined in future experiments. Our work reveals the possibility of rich novel physics in multilayer superconductors with interlayer pairing.

INTRODUCTION

The recent discoveries of superconductivity in bilayer^{1–3} and trilayer^{4–8} nickelates under high pressure have stimulated intensive experimental^{9–21} and theoretical^{22–73} debates concerning their pairing mechanisms as well as potential connections to the cuprate high-temperature superconductivity. While in-plane pairing has been well established in cuprates, both interlayer and intralayer pairing scenarios have been proposed to explain superconductivity in bilayer and trilayer nickelates. Latest resonant inelastic X-ray scattering (RIXS) measurements on $\text{La}_3\text{Ni}_2\text{O}_7$ at ambient pressure¹⁷ reported a dominant interlayer superexchange interaction J , which seems to support the interlayer scenario. However, straightforward experimental confirmation remains challenging due to the high pressure.

In this work, we point out that the different variation of the maximum T_c in multilayer cuprate and nickelate superconductors might be an indicator of their possibly distinct pairing mechanisms. In cuprates, the maximum T_c is the highest in trilayer systems. At ambient pressure, it rises from 97 K (single layer) to 135 K (trilayer) in Hg-based cuprates and from 30 K to 115 K in Bi-based cuprates.^{74,75} In contrast, the maximum T_c is reduced from about 80 K in the bilayer nickelate $\text{La}_3\text{Ni}_2\text{O}_7$ to about 30 K in the trilayer nickelate $\text{La}_4\text{Ni}_3\text{O}_{10}$. To understand this difference, we propose an interlayer pairing scenario and perform numerical simulations on an effective low-energy model of the trilayer nickelate. Our calculations yield a maximum ratio $T_c/J \approx 0.02 - 0.03$, which is reduced by nearly a half compared to that of $0.04 - 0.05$ in the bilayer model.⁴⁴ This reduction is primarily attributed to strong superconducting fluctuations induced by some intrinsic frustration of the interlayer pairing fields, which is absent in intralayer pairing, while imbalance between the inner and two outer layers may further suppress T_c in the trilayer structure. An intrinsic Josephson coupling is also revealed that may have potentially interesting implications for future experimental investigations.

MATERIALS AND METHODS

We start with the following low-energy effective two-orbital trilayer model based on experimental and first-principles considerations,^{10,72,76}

$$H = - \sum_{ij} (t_{ij} + \mu \delta_{ij}) c_{i\sigma}^\dagger c_{j\sigma} - \sum_{ij} V_{ij} (c_{i\sigma}^\dagger d_{j\sigma} + h.c.) - t_\perp \sum_{ais} (d_{ois}^\dagger d_{ais} + h.c.) + J \sum_{ai} S_{oi} \cdot S_{ai}, \quad (1)$$

where d_{is} (c_{is}) is the annihilation operator of d_{z^2} ($d_{x^2-y^2}$) electrons with spin s on site i of the inner ($i=0$) or two outer ($i=\pm$) layers, $S_i = \frac{1}{2} \sum_{\sigma\sigma'} d_{i\sigma}^\dagger \boldsymbol{\sigma}_{\sigma\sigma'} d_{i\sigma'}$ is the d_{z^2} spin density operator with $\boldsymbol{\sigma}$ being the Pauli matrices, the subscript $a=\pm$ denotes two outer layers, t_{ij} and μ are the in-plane nearest-neighbor hopping and the chemical potential of $d_{x^2-y^2}$, V_{ij} is the renormalized in-plane hybridization between nearest-neighbor d_{z^2} and $d_{x^2-y^2}$ orbitals satisfying $V_{ij+\hat{x}} = -V_{ij+\hat{y}} = V$, $t_\perp$ is the renormalized interlayer hopping of d_{z^2} quasiparticles, and J is the d_{z^2} interlayer superexchange interaction mediated by apical oxygens. Other small parameters are ignored for simplicity. The effect of the Hund's rule coupling is incorporated into the renormalized hybridization V which is set as a free tuning parameter.^{32,40} The intralayer superexchange interaction is small as reported in latest RIXS and inelastic neutron scattering (INS) measurements,^{16,17} and thus ignored to focus on interlayer pairing only.

We employ the static auxiliary field Monte Carlo approach, which has been widely applied to strongly correlated electron studies.^{77–88} Its application to the bilayer nickelate superconductor has successfully predicted the evolution of T_c in good consistency with experiment.⁴⁴ To simulate the trilayer model, we decouple the superexchange term:⁸⁹

$$JS_{oi} \cdot S_{ai} \rightarrow \sqrt{2} \bar{\Delta}_i^a \Phi_i^a + \sqrt{2} \bar{\Phi}_i^a \Delta_i^a + \frac{8 \bar{\Delta}_i^a \Delta_i^a}{3J}, \quad (2)$$

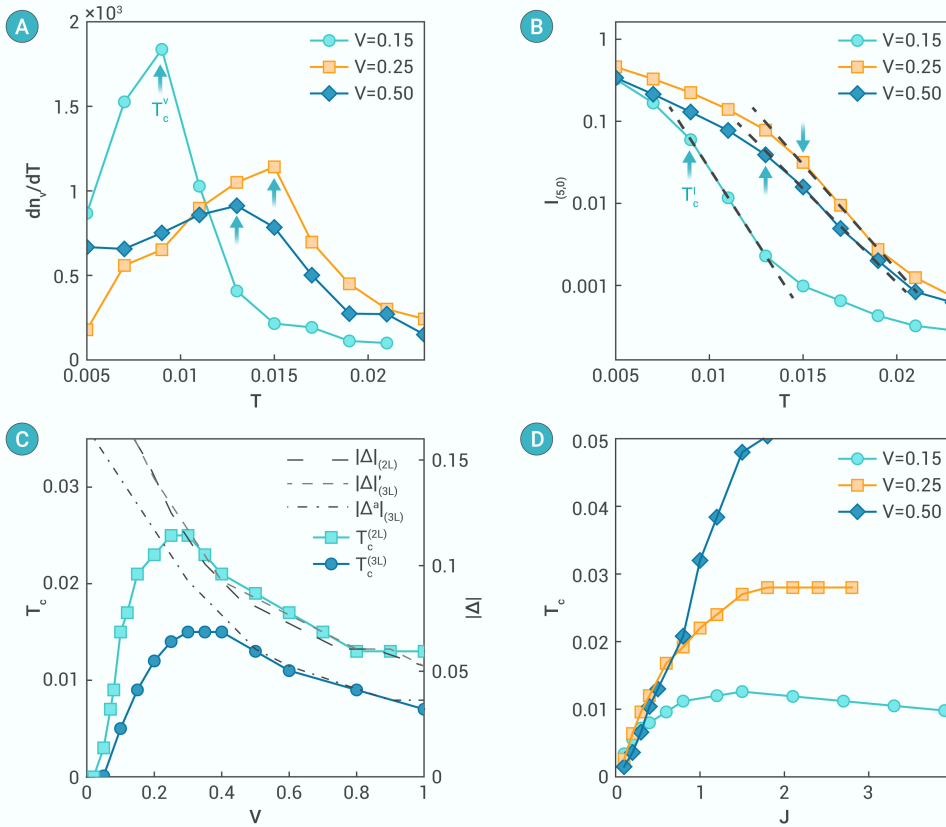
where $\Phi_i^a = \frac{1}{\sqrt{2}} (d_{oi} d_{ai\uparrow} - d_{oi\downarrow} d_{ai\downarrow})$ denotes the local interlayer d_{z^2} spin singlet at site i between the inner and a -th outer layers, and Δ_i^a is its corresponding pairing field. The static approximation ignores the imaginary time dependence of the auxiliary fields, $\Delta_i^a(\tau) \rightarrow \Delta_i^a = |\Delta_i^a| e^{i\theta_i^a}$, to avoid the severe sign problem, while the full thermodynamic fluctuations are retained beyond the uniform mean-field approximation. Hence, it is particularly suitable for studying the Berezinskii-Kosterlitz-Thouless (BKT) transition in two-dimensional superconductivity at finite temperatures.^{90–92}

We first integrate out all fermionic degrees of freedom to derive an effective action that solely depends on the pairing fields:

$$S_{\text{eff}}(\{\Delta_i^a\}) = \frac{8\beta}{3J} \sum_i \bar{\Delta}_i^a \Delta_i^a - \sum_i \ln(1 + e^{-\beta \bar{\Delta}_i^a}) - \sum_i \ln(1 + e^{-\beta \Delta_i^a}), \quad (3)$$

where $\bar{\Delta}_i$ and Δ_i are the eigenvalues of two matrices determined by the tight-binding parameters and the pairing configuration $\{\Delta_i^a\}$ (see supplemental information). Monte Carlo simulations are then performed to sample the probability distribution $p(\{\Delta_i^a\}) = Z^{-1} e^{-S_{\text{eff}}(\{\Delta_i^a\})}$, where Z is the partition function serving as a normalization factor. Hereafter, we will set t to unity as the

Figure 1. Superconducting transition temperature Temperature evolution of (A) the derivative dn_v/dT of the vortex number n_v and (B) the superconducting phase mutual information $I_{(5,0)}$ for three typical values of the hybridization parameter $V = 0.15, 0.25, 0.50$ and $J = 0.5$. The maximum of dn_v/dT reflects the BKT transition for two-dimensional superconductivity and defines the temperature scale T_v^* , and the slope change (dashed lines) in $I_{(5,0)}$ at low temperatures marks the long-distance phase coherence and defines the temperature scale T_c^* . They together define the superconducting transition temperature T_c . (C) Comparison of T_c for trilayer and bilayer models as functions of the hybridization V at $J = 0.5$. Also shown are the most probable values of the pairing amplitudes Δ^a and $|\Delta|' = \sqrt{|\Delta^+|^2 + |\Delta^-|^2}$ of the trilayer model and $|\Delta|$ of the bilayer model. (D) Evolution of the trilayer T_c as functions of the interlayer superexchange interaction J for $V = 0.15, 0.25, 0.50$.



energy unit and fix the chemical potential $\mu = -1.3$ to ensure that the total electron density varies only slightly around its nominal value 2.67. Due to computational limitations, all results presented are obtained on a 10×10 trilayer lattice but the conclusions have been examined to be robust across other lattice sizes.

RESULTS

Reduction of T_c

Theoretically, the BKT transition is characterized by a rapid increase of the vortex number n_v .⁹³ Figure 1A plots its temperature derivative dn_v/dT for three different values of V at fixed $J = 0.5$ and $t_\perp = 0$. A peak is clearly seen for each curve that defines the temperature scale T_v^* . To confirm that this indeed marks a superconducting transition, we further calculate the phase mutual information $I_{\mathbf{R}}$ between two local pairing fields of the largest distance $\mathbf{R} = (5, 0)$ along the x direction.^{44,94–99} As shown in Figure 1B, the phase mutual information displays several slope changes with lowering temperature in the semilog plot. Its rapid increase at intermediate temperature region marks the development of long-distance phase correlation, with a lower boundary T_c coinciding with T_v^* determined from the vortex number. We thus identify them as the superconducting transition temperature T_c at which phase coherence is developed between local pairs on distant interlayer bonds.

Figure 1C compares the estimated T_c for bilayer and trilayer models of the same parameters. Both vary nonmonotonically with the hybridization parameter V and reach a maximum at around $V = 0.3$. The nonmonotonic variation reflects the competition between in-plane phase coherence induced by the hybridization V and local interlayer pairing determined mainly by the superexchange J . We find T_c to be greatly reduced from bilayer to trilayer models. This observation is further supported by Figure 1D where the trilayer T_c is plotted against J for three typical values of $V = 0.15, 0.25, 0.50$. We obtain the largest slope in the small J region, giving a maximum ratio $T_c/J \approx 0.02 - 0.03$ for realistic parameters. While for the bilayer model, our previous calculations yield a maximum $T_c/J \approx 0.04 - 0.05$.⁴⁴ Interestingly, if we take heuristically the interlayer J to be 150 meV from RIXS measurements (assuming the spin size $S \approx 1/2$) on $\text{La}_3\text{Ni}_2\text{O}_7$ at ambient pressure,¹⁷ these ratios immediately predict a maximum T_c of about 70–90 K for the bilayer system and 30–50 K for the trilayer system, both consistent with experimental observations.^{1,7} In contrast,

the intralayer J is less than 10 meV at ambient pressure, whereas in cuprates it is much larger than the interlayer one.¹⁰⁰

In reality, the magnitude of the trilayer T_c may be further reduced by other factors. First-principles calculations have revealed a subtle imbalance in $\text{La}_3\text{Ni}_3\text{O}_{10}$, with somewhat different hybridization V and chemical potential μ between the inner and outer layers.^{66,72} Our calculations show that both tend to suppress T_c (see supplemental information). A small interlayer hopping may also influence T_c but acts

similarly on both models. We therefore conclude that the reduction of T_c from bilayer to trilayer nickelates may be a genuine intrinsic property reflecting their interlayer pairing mechanism, different from the cuprates.

Frustrated superconductivity

To clarify the origin of the T_c reduction, we present a comparative analysis of the probability distribution of the pairing field amplitudes in bilayer and trilayer models. As shown in Figure 2A, the trilayer $p(|\Delta^a|)$ exhibits a broad peak extending almost linearly to $|\Delta^a| = 0$, in sharp contrast to that of the bilayer model. To understand this, we collect all Monte Carlo configurations of $|\Delta^a|$ of the trilayer model on an intensity plot in Figure 2C for two typical values $V = 0.15$ and 0.5 . Quite unexpectedly, the data accumulate to form a ring shape at low temperatures. As the temperature increases, the ring gradually dissolves, resulting in nearly independent fluctuations of $|\Delta^+|$ and $|\Delta^-|$. Note that the suppression of the distribution near horizontal and vertical axes is due to the phase space limitation of the amplitude. For small $V = 0.15$, the ring remains discernible even above T_c due to preformed pairs in this parameter region.^{101,102}

The ring shape suggests some internal symmetry between $|\Delta^+|$ and motivates us to define an overall pairing amplitude $|\Delta|' \equiv \sqrt{|\Delta^+|^2 + |\Delta^-|^2}$. Interestingly, as shown in Figure 2B, the trilayer $p(|\Delta|')$ exhibits almost identical distribution as the bilayer $p(|\Delta|)$. To see this more clearly, we plot in Figure 1C the most probable value of $|\Delta^a|$ and $|\Delta|'$ of the trilayer model and $|\Delta|$ of the bilayer model, and compare them with T_c after a common scaling. Surprisingly, $|\Delta|'$ and $|\Delta|$ fall upon the same curve and track exactly the T_c evolution of the bilayer model at large V , while $|\Delta^a|$ is systematically smaller and falls upon the T_c curve of the trilayer model. Thus, the magnitudes of T_c in this region are determined by the pairing amplitudes in both models, and its reduction in the trilayer model must originate from the fluctuation between $|\Delta^+|$ and $|\Delta^-|$. The superconductivity seems frustrated, where two outer layers compete to form the singlet with the inner layer to achieve an overall $|\Delta|' \approx |\Delta|$.

Internal symmetry and Josephson coupling

To clarify the origin of the frustration, we expand the effective action up to

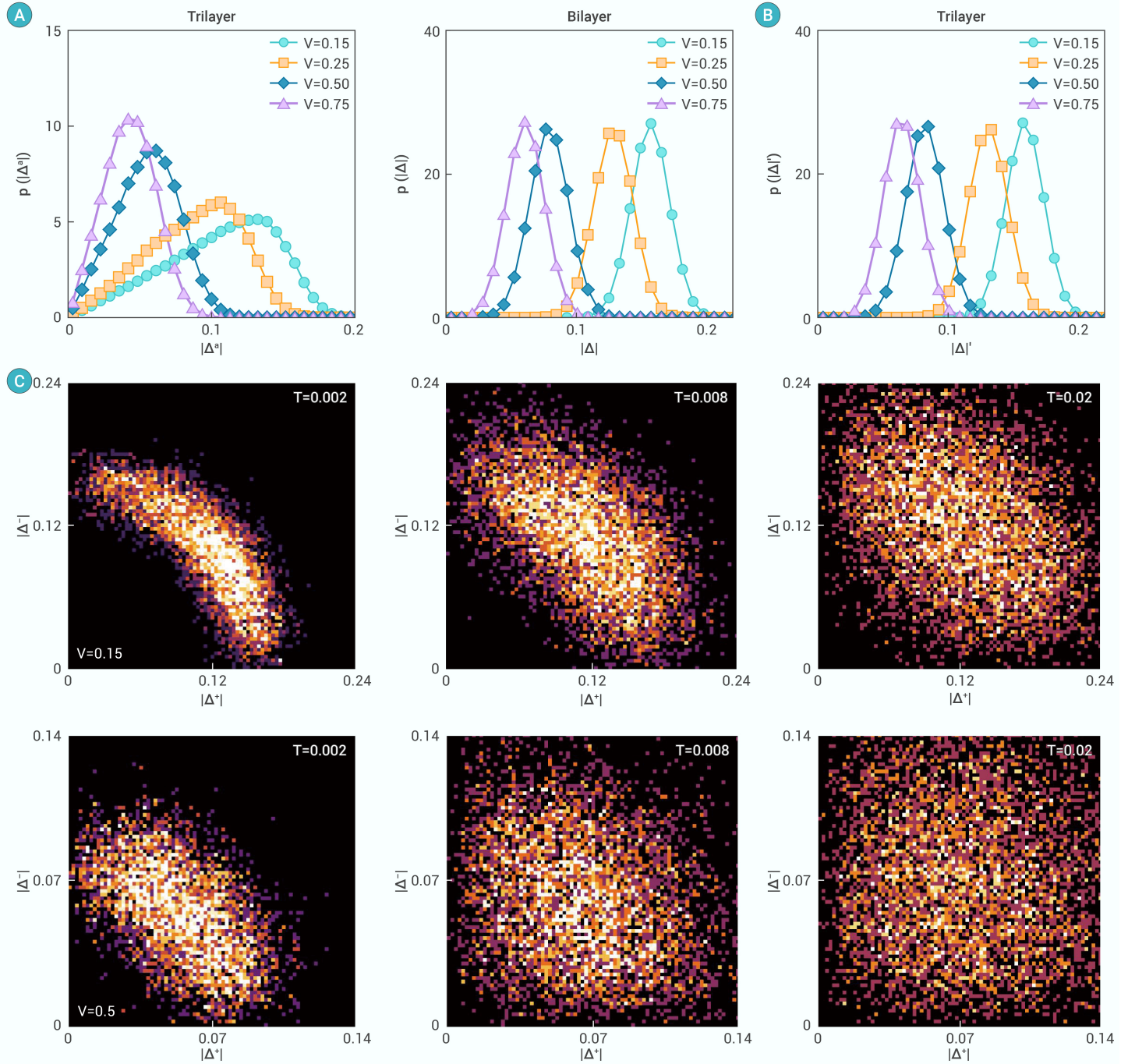


Figure 2. Interlayer superconducting fluctuations (A) Comparison of the probability distributions of the pairing amplitudes $p(|\Delta^+|)$ of the trilayer model (left) and $p(|\Delta|)$ of the bilayer model (right) for different values of $V = 0.15, 0.25, 0.50, 0.75$ at $T = 0.002$. (B) Probability distribution of the total pairing amplitude $p(|\Delta|')$ of the trilayer model, where $|\Delta|' = \sqrt{|\Delta^+|^2 + |\Delta^-|^2}$. (C) Intensity plot of the joint distribution of $|\Delta^+|$ and $|\Delta^-|$ of the trilayer model for $V = 0.15, 0.5$ and $T = 0.002, 0.008, 0.02$, showing clear ring-shaped distribution at low temperatures and weak hybridizations. Other parameters are $t_{\perp} = 0$ and $J = 0.5$.

fourth order by assuming uniform pairing fields for simplicity. This gives a Ginzburg-Landau (GL) free energy for the trilayer model:

$$f_{\text{GL}} = c_1 \psi^\dagger \psi + c_2 (\psi^\dagger \psi)^2 - 2h \mathcal{T}_x^A, \quad (4)$$

where $\psi = (\Delta^+, \Delta^-)^T$ is a Higgs-like complex doublet order parameter, $\mathcal{T}^A = \frac{1}{2} \psi^\dagger \boldsymbol{\sigma} \psi$ is its pseudospin, $c_{1/2}$ are two GL parameters determined by model details, and $h \propto t_{\perp}^2$ plays the role of an effective transverse field. Detailed derivations can be found in the supplemental information.

We first consider $t_{\perp} = 0$ since it is strongly renormalized and proportional to the density of self-doped holes on d_{z^2} ($\delta_h \propto 0.01$). This gives $h = 0$ and we immediately see that the GL free energy has a global $\text{SU}(2) \times \text{U}(1)$ symmetry

associated with the rotation of the pseudospin and an overall phase rotation. Both symmetries can be well understood from the original Hamiltonian (1), where the $\text{SU}(2)$ corresponds to the basis transformation of $(d_{+is}, d_{-is})^T$ and $(c_{+is}, c_{-is})^T$ between two outer layers, and the $\text{U}(1)$ reflects their phase change relative to the inner layer (charge conservation). For $c_1 < 0$ and $c_2 > 0$, $\psi^\dagger \psi = |\Delta^+|^2 + |\Delta^-|^2$ acquires a nonzero expectation $-c_1/2c_2$, indicating a finite mean-field value of the total amplitude $|\Delta|'$. However, the Mermin-Wagner theorem forbids the spontaneous continuous symmetry breaking at finite temperature, causing the ring-shaped joint distribution of $|\Delta^+|$ and $|\Delta^-|$ observed in Figure 2C. For the same reason, there is no phase coherence between two superconducting layers, since $\langle e^{-i(\theta^+ - \theta^-)} \rangle \propto \langle \Delta^+ \Delta^- \rangle =$

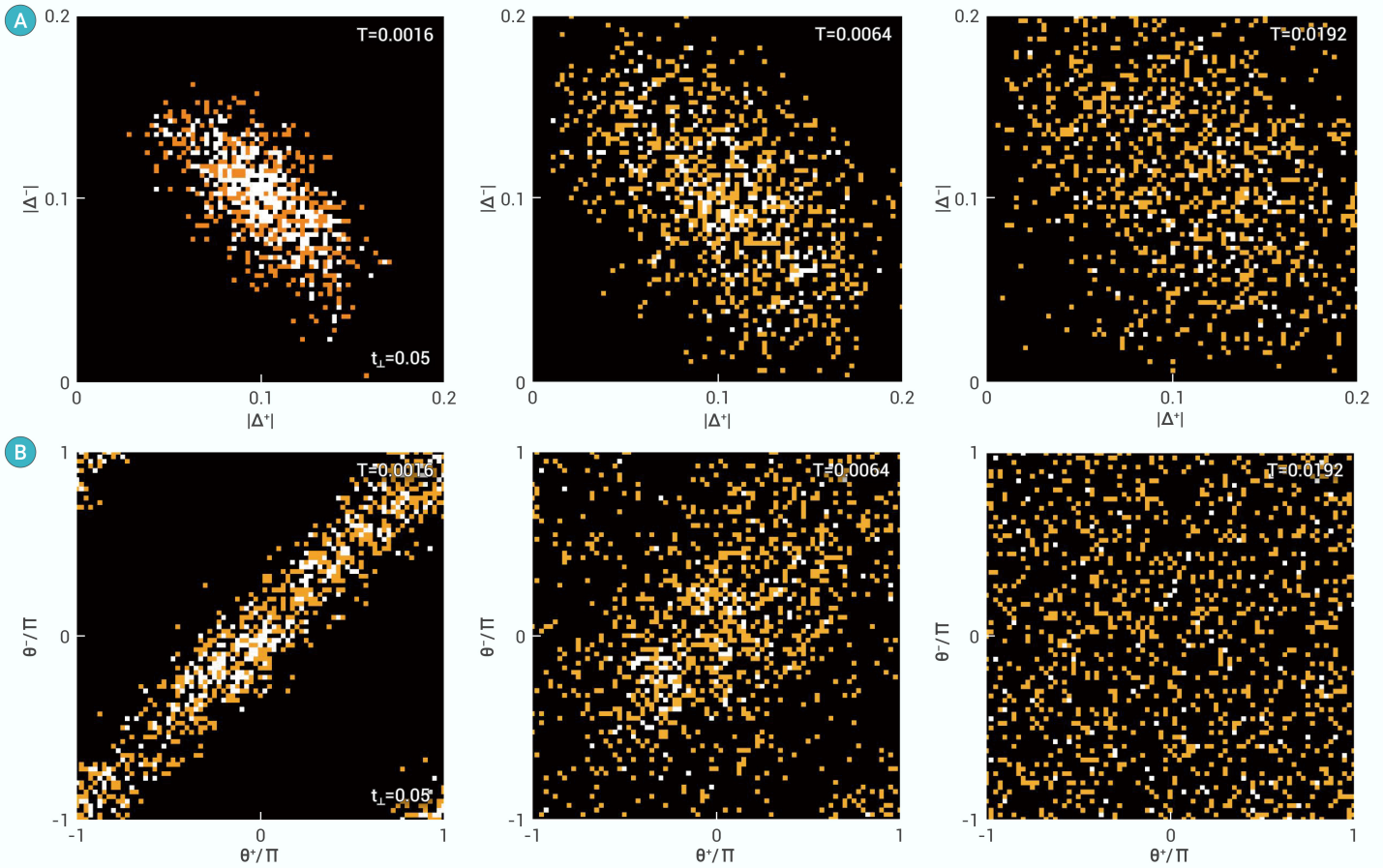


Figure 3. Effect of the interlayer hopping Intensity plot of the joint distribution of (A) the pairing amplitudes $|\Delta^\pm|$ and (B) their phases $\theta^\pm$ of the trilayer model, showing strong fluctuations even deep inside the superconductivity for a finite $t_\perp = 0.05$. Other parameters are $V = 0.15$ and $J = 0.5$, with $T_c \approx 0.008$ reduced slightly from 0.009 at $t_\perp = 0$.

$\langle \mathcal{J}_x^A \rangle + i \langle \mathcal{J}_y^A \rangle = 0$, which is also observed in our simulations. The free energy has the same form for the bilayer $|\Delta|$, explaining their almost identical distributions.

A nonzero interlayer hopping introduces a small but finite $h \propto t_\perp^2 \propto \delta_h^2$, which plays the role of a transverse field in the GL free energy and breaks the global SU(2) symmetry. For $h > 0 (< 0)$, the GL free energy is minimized at $\theta^+ = \theta^-$ ($\theta^+ - \theta^- = \pi$) and $|\Delta^+| = |\Delta^-|$, with Gaussian fluctuations of both $|\Delta^+| - |\Delta^-|$ and $\theta^+ - \theta^-$ proportional to δ_h^{-1} , as derived in the supplemental information. Thus, the fluctuations remain strong for small δ_h . This is confirmed in Figures 3A-B from our numerical calculations of the joint distribution of $|\Delta^\pm|$ and $\theta^\pm$ for $t_\perp = 0.05$ and $V = 0.15$. Indeed, while the data get more clustered around the GL solution with lowering temperature, strong fluctuations persist even at $T = 0.0016$ well below the BKT transition, indicating the lack of coherence along the c -axis. By rewriting $\mathcal{J}_x^A = |\Delta^+ \Delta^-| \cos(\theta^+ - \theta^-)$, we find an intrinsic Josephson coupling within the trilayer structure, which predicts a Josephson frequency (if exist) only half that for intralayer pairing but is absent in the bilayer model (see supplemental information). We suggest future experiments to examine to what extent these features may be realized in real materials.

Pairing symmetry

The mean-field gap structures in momentum space may be tentatively obtained by constructing a Bardeen-Cooper-Schrieffer (BCS) Hamiltonian with the static pairing fields Δ^a and realistic tight-binding parameters.⁷² First-principles calculations have predicted four Fermi pockets denoted as α , $\beta_{1/2}$, and γ in Figure 4A. Among them, γ comes mainly from the d_{z^2} bonding orbital, and all three others are hybridization bands of both d_{z^2} and $d_{x^2-y^2}$ characters. One may roughly think of the two outer layers to first form bonding and antibonding orbitals. Their antibonding orbitals give rise to the β_2 Fermi surface, while their bonding orbitals further hybridize with the inner layer to

form a bonding combination (α) and an antibonding combination (β_1). Such bonding/antibonding characters determine most of their gap properties except for some narrow regions. The BCS Hamiltonian can be diagonalized to project out the gaps on each Fermi pocket. Figures 4A-B plot some typical results and we find extended $s^\pm$ -wave gaps with (accidental) nodes along and close to the zone diagonal. Specifically, the gap on γ is nodeless, the gap on α has the same sign as γ but contains nodes along the zone diagonal where the $d_{x^2-y^2} - d_{z^2}$ hybridization is zero, and the gap on β_1 has an opposite sign on most regions of the Fermi surface with nodes along the zone diagonal but, depending on model details, may also change sign to give some accidental nodes near the zone diagonal. Notably, β_2 remains metallic because of its antibonding character formed of two outer layers. Some of these details may change if other pairing channels are included. For example, introducing additional interlayer $d_{x^2-y^2}$ pairing might eliminate the sign change and accidental nodes on β_1 , while direct interlayer pairing of d_{z^2} between two outer layers would open a superconducting gap on β_2 . But the main features, namely the extended $s^\pm$ -wave pairing symmetry and the gap nodes (or minima) along the zone diagonal, are robust as long as the d_{z^2} interlayer pairing interaction dominates.

DISCUSSION

We have proposed an interlayer pairing scenario for $\text{La}_4\text{Ni}_3\text{O}_{10}$ and derived a reduced maximum ratio $T_c/J \approx 0.02 - 0.03$ in the trilayer model from 0.04 - 0.05 in the bilayer model. We show that this reduction may be a natural consequence of interlayer pairing due to the competition of two outer layers to form spin-singlet pairs with the inner layer respectively. This induces some kind of frustration and enhances the superconducting fluctuations between layers. Similar simulations for intralayer pairing have been performed for cuprates, but the maximum ratio T_c/J seems robust except for some very special situations.¹⁰³ It has been argued that the observed enhancement of

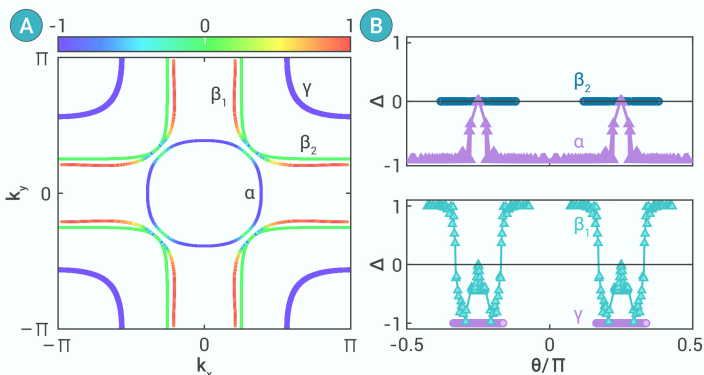


Figure 4. Superconducting gap structures (A) Illustration of some typical gap structures (normalized to their maximum values) on four Fermi pockets predicted by first-principles calculations. (B) Their variations with the azimuthal angle θ within $(-\pi/2, \pi/2)$.

maximum T_c in multilayer cuprates might actually come from the increase of the intralayer superexchange interaction or tuning to reach the optimal condition for maximizing T_c/J . While such details may be debated, our systematic comparison here reveals a probably fundamental distinction between two systems associated with their potentially different (interlayer versus intralayer) pairing mechanisms. An alternative explanation would be demanded if intralayer pairing governs bilayer and trilayer nickelate superconductors.

Lately, it has been reported that $\text{La}_3\text{Ni}_2\text{O}_7$ may also adopt a single layer-trilayer alternate (1313) structure,^{104–106} raising concerns regarding the crystal structure truly responsible for its high-temperature superconductivity. While some recent experiments have already provided supportive evidences for the bilayer structure,¹⁰⁷ our calculations here imply that the (1313) structure cannot give a maximum T_c as high as that in the bilayer structure.

All together, our work reveals potentially rich physics in multilayer superconductors with interlayer pairing, which may be distinctively different from that for intralayer pairing but has not been well noticed before. This points out a promising novel direction for future investigations.

REFERENCES

- Sun, H., Huo, M., Hu, X., et al. (2023). Signatures of superconductivity near 80 K in a nickelate under high pressure. *Nature* **621**: 493–498. DOI: 10.1038/s41586-023-06408-7.
- Hou, J., Yang, P.-T., Liu, Z.-Y., et al. (2023). Emergence of high-temperature superconducting phase in the pressurized $\text{La}_3\text{Ni}_2\text{O}_7$ crystals. *Chin. Phys. Lett.* **40**: 117302. DOI: 10.1088/0256-307X/40/11/117302.
- Zhang, Y., Su, D., Huang, Y. et al. (2024). High-temperature superconductivity with zero resistance and strange-metal behaviour in $\text{La}_3\text{Ni}_2\text{O}_{7-\delta}$. *Nat. Phys.* **20**: 1269–1273. DOI: 10.1038/s41567-024-02515-y.
- Sakakibara, H., Ochi, M., Nagata, H., et al. (2024). Theoretical analysis on the possibility of superconductivity in the trilayer Ruddlesden-Popper nickelate $\text{La}_4\text{Ni}_3\text{O}_{10}$ under pressure and its experimental examination: Comparison with $\text{La}_3\text{Ni}_2\text{O}_7$. *Phys. Rev. B* **109**: 144511. DOI: 10.1103/PhysRevB.109.144511.
- Li, Q., Zhang, Y.-J., Xiang, Z.-N., et al. (2024). Signature of superconductivity in pressurized $\text{La}_4\text{Ni}_3\text{O}_{10}$. *Chin. Phys. Lett.* **41**: 017401. DOI: 10.1088/0256-307X/41/1/017401.
- Zhu, Y., Peng, D., Zhang, E. et al. (2024). Superconductivity in pressurized trilayer $\text{La}_4\text{Ni}_3\text{O}_{10-\delta}$ single crystals. *Nature* **631**: 531–536. DOI: 10.1038/s41586-024-07553-3.
- Zhang, M., Pei, C., Du, X., et al. (2023). Superconductivity in trilayer nickelate $\text{La}_4\text{Ni}_3\text{O}_{10}$ under pressure. *arXiv:2311.07423*. DOI: 10.48550/arXiv.2311.07423.
- Li, J., Chen, C.-Q., Huang, C., et al. (2024). Structural transition, electric transport, and electronic structures in the compressed trilayer nickelate $\text{La}_4\text{Ni}_3\text{O}_{10}$. *Sci. China-Phys. Mech. Astron.* **67**: 117403. DOI: 10.1007/s11433-023-2329-x.
- Liu, Z., Huo, M., Li, J. et al. (2024). Electronic correlations and partial gap in the bilayer nickelate $\text{La}_3\text{Ni}_2\text{O}_7$. *Nat. Commun.* **15**: 7570. DOI: 10.1038/s41467-024-52001-5.
- Yang, J., Sun, H., Hu, X. et al. (2024). Orbital-dependent electron correlation in double-layer nickelate $\text{La}_3\text{Ni}_2\text{O}_7$. *Nat. Commun.* **15**: 4373. DOI: 10.1038/s41467-024-48701-7.
- Wang, G., Wang, N.-N., Shen, X.-L., et al. (2024). Pressure-induced superconductivity

in polycrystalline $\text{La}_3\text{Ni}_2\text{O}_{7-\delta}$. *Phys. Rev. X* **14**: 011040. DOI: 10.1103/PhysRevX.14.011040.

- Zhou, Y., Guo, J., Cai, S., et al. (2023). Evidence of filamentary superconductivity in pressurized $\text{La}_3\text{Ni}_2\text{O}_7$. *arXiv:2311.12361*. DOI: 10.48550/arXiv.2311.12361.
- Cui, T., Choi, S., Lin, T. et al. (2024). Strain-mediated phase crossover in Ruddlesden–Popper nickelates. *Commun. Mater.* **5**: 32. DOI: 10.1038/s43246-024-00478-4.
- Chen, K., Liu, X., Jiao, J., et al. (2024). Evidence of spin density waves in $\text{La}_3\text{Ni}_2\text{O}_{7-\delta}$. *Phys. Rev. Lett.* **132**: 256503. DOI: 10.1103/PhysRevLett.132.256503.
- Dong, Z., Huo, M., Li, J. et al. (2024). Visualization of oxygen vacancies and self-doped ligand holes in $\text{La}_3\text{Ni}_2\text{O}_{7-\delta}$. *Nature* **630**: 847–852. DOI: 10.1038/s41586-024-07482-1.
- Xie, T., Huo, M., Ni, X., et al. (2024). Neutron scattering studies on the high- T_c superconductor $\text{La}_3\text{Ni}_2\text{O}_{7-\delta}$ at ambient pressure. *arXiv:2401.12635*. DOI: 10.48550/arXiv.2401.12635.
- Chen, X., Choi, J., Jiang, Z., et al. (2024). Electronic and magnetic excitations in $\text{La}_3\text{Ni}_2\text{O}_7$. *arXiv:2401.12657*. DOI: 10.48550/arXiv.2401.12657.
- Dan, Z., Zhou, Y., Huo, M., et al. (2024). Spin-density-wave transition in double-layer nickelate $\text{La}_3\text{Ni}_2\text{O}_7$. *arXiv:2402.03952*. DOI: 10.48550/arXiv.2402.03952.
- Abadi, S. N., Xu, K. J., Lomeli, E. G., et al. (2024). Electronic structure of the alternating monolayer-trilayer phase of $\text{La}_3\text{Ni}_2\text{O}_7$. *arXiv:2402.07143*. DOI: 10.48550/arXiv.2402.07143.
- Yuan, N., Elghandour, A., Arneth, J., et al. (2024). High-pressure crystal growth and investigation of the metal-to-metal transition of Ruddlesden–Popper trilayer nickelates $\text{La}_4\text{Ni}_3\text{O}_{10}$. *J. Cryst. Growth* **627**: 127511. DOI: 10.1016/j.jcrysgro.2023.127511.
- Kakoi, M., Oi, T., Ohshita, Y., et al. (2024). Multiband metallic ground state in multilayered nickelates $\text{La}_3\text{Ni}_2\text{O}_7$ and $\text{La}_4\text{Ni}_3\text{O}_{10}$ Probed by ^{139}La -NMR at Ambient Pressure. *J. Phys. Soc. Jpn.* **93**: 053702. DOI: 10.7566/JPSJ.93.053702.
- Luo, Z., Hu, X., Wang, M., et al. (2023). Bilayer two-orbital model of $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. *Phys. Rev. Lett.* **131**: 126001. DOI: 10.1103/PhysRevLett.131.126001.
- Zhang, Y., Lin, L.-F., Moreo, A., et al. (2023). Electronic structure, dimer physics, orbital-selective behavior, and magnetic tendencies in the bilayer nickelate superconductor $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. *Phys. Rev. B* **108**: L180510. DOI: 10.1103/PhysRevB.108.L180510.
- Yang, Q.-G., Wang, D., and Wang, Q.-H. (2023). Possible s_x -wave superconductivity in $\text{La}_3\text{Ni}_2\text{O}_7$. *Phys. Rev. B* **108**: L140505. DOI: 10.1103/PhysRevB.108.L140505.
- Lechermann, F., Gondolf, J., Bötzel, S., et al. (2023). Electronic correlations and superconducting instability in $\text{La}_3\text{Ni}_2\text{O}_7$ under high pressure. *Phys. Rev. B* **108**: L201121. DOI: 10.1103/PhysRevB.108.L201121.
- Sakakibara, H., Kitamine, N., Ochi, M., et al. (2024). Possible high T_c superconductivity in $\text{La}_3\text{Ni}_2\text{O}_7$ under high pressure through manifestation of a nearly-half-filled bilayer Hubbard model. *Phys. Rev. Lett.* **132**: 106002. DOI: 10.1103/PhysRevLett.132.106002.
- Gu, Y., Le, C., Yang, Z., et al. (2023). Effective model and pairing tendency in bilayer Ni-based superconductor $\text{La}_3\text{Ni}_2\text{O}_7$. *arXiv:2306.07275*. DOI: 10.48550/arXiv.2306.07275.
- Shen, Y., Qin, M., and Zhang, G.-M. (2023). Effective bilayer model hamiltonian and density-matrix renormalization group study for the high- T_c superconductivity in $\text{La}_3\text{Ni}_2\text{O}_7$ under high pressure. *Chin. Phys. Lett.* **40**: 127401. DOI: 10.1088/0256307X/40/12/127401.
- Christiansson, V., Petocchi, F., and Werner, P. (2023). Correlated electronic structure of $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. *Phys. Rev. Lett.* **131**: 206501. DOI: 10.1103/PhysRevLett.131.206501.
- Shilenko, D. A. and Leonov, I. V. (2023). Correlated electronic structure, orbital-selective behavior, and magnetic correlations in double-layer $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. *Phys. Rev. B* **108**: 125105. DOI: 10.1103/PhysRevB.108.125105.
- Wu, W., Luo, Z., Yao, D.-X. et al. (2024). Charge transfer and Zhang-Rice singlet bands in the nickelate superconductor $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. *Sci. China-Phys. Mech. Astron.* **67**: 117402. DOI: 10.1007/s11433-023-2300-4.
- Cao, Y. and Yang, Y.-F. (2024). Flat bands promoted by Hund's rule coupling in the candidate double-layer high-temperature superconductor $\text{La}_3\text{Ni}_2\text{O}_7$. *Phys. Rev. B* **109**: L081105. DOI: 10.1103/PhysRevB.109.L081105.
- Chen, X., Jiang, P., Li, J., et al. (2023). Critical charge and spin instabilities in superconducting $\text{La}_3\text{Ni}_2\text{O}_7$. *arXiv:2307.07154*. DOI: 10.48550/arXiv.2307.07154.
- Liu, Y.-B., Mei, J.-W., Ye, F., et al. (2023). $s^{\pm}$ -Wave pairing and the destructive role of apical-oxygen deficiencies in $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. *Phys. Rev. Lett.* **131**: 236002. DOI: 10.1103/PhysRevLett.131.236002.
- Lu, C., Pan, Z., Yang, F., et al. (2024). Interlayer coupling driven high-temperature superconductivity in $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. *Phys. Rev. Lett.* **132**: 146002. DOI: 10.1103/PhysRevLett.132.146002.
- Zhang, Y., Lin, L.-F., Moreo, A. et al. (2024). Structural phase transition, s_x -wave pairing and magnetic stripe order in the bilayered nickelate superconductor $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. *Nat. Commun.* **15**: 2470. DOI: 10.1038/s41467-024-46622-z.
- Oh, H. and Zhang, Y.-H. (2023). Type-II t - J model and shared superexchange

- coupling from Hund's rule in superconducting $\text{La}_3\text{Ni}_2\text{O}_7$. Phys. Rev. B **108**: 174511. DOI: 10.1103/PhysRevB.108.174511.
38. Liao, Z., Chen, L., Duan, G., et al. (2023). Electron correlations and superconductivity in $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure tuning. Phys. Rev. B **108**: 214522. DOI: 10.1103/PhysRevB.108.214522.
 39. Qu, X.-Z., Qu, D.-W., Chen, J., et al. (2024). Bilayer t - J - $J_\perp$ model and magnetically mediated pairing in the pressurized nickelate $\text{La}_3\text{Ni}_2\text{O}_7$. Phys. Rev. Lett. **132**: 036502. DOI: 10.1103/PhysRevLett.132.036502.
 40. Yang, Y.-F., Zhang, G.-M., and Zhang, F.-C. (2023). Interlayer valence bonds and two-component theory for high- T_c superconductivity of $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. Phys. Rev. B **108**: L201108. DOI: 10.1103/PhysRevB.108.L201108.
 41. Jiang, K., Wang, Z., and Zhang, F.-C. (2024). Impurity and vortex states in the bilayer $\text{La}_3\text{Ni}_2\text{O}_7$. Chin. Phys. Lett. **41**: 017402. DOI: 10.1088/0256-307X/41/1/017402.
 42. Zhang, Y., Lin, L.-F., Moreo, A., et al. (2023). Trends in electronic structures and s_x -wave pairing for the rare-earth series in bilayer nickelate superconductor $R_3\text{Ni}_2\text{O}_7$. Phys. Rev. B **108**: 165141. DOI: 10.1103/PhysRevB.108.165141.
 43. Huang, J., Wang, Z.-D., and Zhou, T. (2023). Impurity and vortex states in the bilayer high-temperature superconductor $\text{La}_3\text{Ni}_2\text{O}_7$. Phys. Rev. B **108**: 174501. DOI: 10.1103/PhysRevB.108.174501.
 44. Qin, Q. and Yang, Y.-F. (2023). High- T_c superconductivity by mobilizing local spin singlets and possible route to higher T_c in pressurized $\text{La}_3\text{Ni}_2\text{O}_7$. Phys. Rev. B **108**: L140504. DOI: 10.1103/PhysRevB.108.L140504.
 45. Tian, Y.-H., Chen, Y., Wang, J.-M., et al. (2024). Correlation effects and concomitant two-orbital s_x -wave superconductivity in $\text{La}_3\text{Ni}_2\text{O}_7$ under high pressure. Phys. Rev. B **109**: 165154. DOI: 10.1103/PhysRevB.109.165154.
 46. Luo, Z., Lv, B., Wang, M., (2024). High T_c superconductivity in $\text{La}_3\text{Ni}_2\text{O}_7$ based on the bilayer two-orbital t - J model. Npj Quantum Mater. **9**: 61. DOI: 10.1038/s41535-024-00668-w.
 47. Zhang, J.-X., Zhang, H.-K., You, Y.-Z., et al. (2024). Strong pairing originated from an emergent Z_2 Berry phase in $\text{La}_3\text{Ni}_2\text{O}_7$. Phys. Rev. Lett. **133**: 126501 (2024). DOI: 10.1103/PhysRevLett.133.126501.
 48. Geisler, B., Hamlin, J. J., Stewart, G. R., et al. (2024). Structural transitions, octahedral rotations, and electronic properties of $\text{A}_3\text{Ni}_2\text{O}_7$ rare-earth nickelates under high pressure. Npj Quantum Mater. **9**: 38. DOI: 10.1038/s41535-024-00648-0.
 49. Kaneko, T., Sakakibara, H., Ochi, M., et al. (2024). Pair correlations in the two-orbital Hubbard ladder: Implications for superconductivity in the bilayer nickelate $\text{La}_3\text{Ni}_2\text{O}_7$. Phys. Rev. B **109**: 045154. DOI: 10.1103/PhysRevB.109.045154.
 50. Lu, C., Pan, Z., Yang, F., et al. (2024). Interplay of two E_g orbitals in superconducting $\text{La}_3\text{Ni}_2\text{O}_7$ under pressure. Phys. Rev. B **110**: 094509. DOI: 10.1103/PhysRevB.110.094509.
 51. Ryee, S., Witt, N., and Wehling, T. O. (2024). Quenched pair breaking by interlayer correlations as a key to superconductivity $\text{La}_3\text{Ni}_2\text{O}_7$. Phys. Rev. Lett. **133**: 096002. DOI: 10.1103/PhysRevLett.133.096002.
 52. Chen, J., Yang, F., and Li, W. (2024). Orbital-selective superconductivity in the pressurized bilayer nickelate $\text{La}_3\text{Ni}_2\text{O}_7$: An infinite projected entangled-pair state study. Phys. Rev. B **110**: L041111. DOI: 10.1103/PhysRevB.110.L041111.
 53. Liu, H., Xia, C., Zhou, S., et al. (2023). Role of crystal-field-splitting and long-range-hoppings on superconducting pairing symmetry of $\text{La}_3\text{Ni}_2\text{O}_7$. arXiv:2311.07316. DOI: 10.48550/arXiv.2311.07316.
 54. Wang, L., Li, Y., Xie, S., et al. (2023). Structure responsible for the superconducting state in $\text{La}_3\text{Ni}_2\text{O}_7$ at high pressure and low temperature conditions. arXiv:2311.09186. DOI: 10.48550/arXiv.2311.09186.
 55. Chang, W.-X., Guo, S., You, Y.-Z. et al. (2023). Fermi surface symmetric mass generation: a quantum Monte-Carlo study. arXiv:2311.09970. DOI: 10.48550/arXiv.2311.09970.
 56. Ouyang, Z., Wang, J.-M., Wang, J.-X., et al. (2024). Hund electronic correlation in $\text{La}_3\text{Ni}_2\text{O}_7$ under high pressure. Phys. Rev. B **109**: 115114. DOI: 10.1103/PhysRevB.109.115114.
 57. Qu, X.-Z., Qu, D.-W., Li, W., et al. (2023). Roles of Hund's rule and hybridization in the two-orbital model for high- T_c superconductivity in the bilayer nickelate. arXiv:2311.12769. DOI: 10.48550/arXiv.2311.12769.
 58. Zheng, Y.-Y. and Wu, W. (2023). Superconductivity in the bilayer two-orbital Hubbard model. arXiv:2312.03605. DOI: 10.48550/arXiv.2312.03605.
 59. Heier, G., Park, K., and Savrasov, S. Y. (2024). Competing d_{xy} and s_x pairing symmetries in superconducting $\text{La}_3\text{Ni}_2\text{O}_7$ emerge from LDA+FLEX calculations. Phys. Rev. B **109**: 104508. DOI: 10.1103/PhysRevB.109.104508.
 60. Fan, Z., Zhang, J.-F., Zhan, B., et al. (2024). Superconductivity in nickelate and cuprate superconductors with strong bilayer coupling. Phys. Rev. B **110**: 024514. DOI: 10.1103/PhysRevB.110.024514.
 61. Talantsev, E. F., and Chistyakov, V. V. (2024). Debye temperature, electron-phonon coupling constant, and microcrystalline strain in highly-compressed $\text{La}_3\text{Ni}_2\text{O}_{7-\delta}$. Lett. on Mater. **14**: 262-268 (2024). DOI: 10.48612/letters/2024-3-262-268.
 62. Wang, Y., Jiang, K., Wang, Z., et al. (2024). Electronic structure and superconductivity in bilayer $\text{La}_3\text{Ni}_2\text{O}_7$. arXiv:2401.15097. DOI: 10.48550/arXiv.2401.15097.
 63. Bötzel, S., Lechermann, F., Gondolf, J., et al. (2024). Theory of magnetic excitations in multilayer nickelate superconductor $\text{La}_3\text{Ni}_2\text{O}_7$. Phys. Rev. B **109**: L180502. DOI: 10.1103/PhysRevB.109.L180502.
 64. Xue, J.-R., and Wang, F. (2024). Magnetism and superconductivity in the t - J model of $\text{La}_3\text{Ni}_2\text{O}_7$ under multiband gutzwiller approximation. Chin. Phys. Lett. **41**: 05740. DOI: 10.1088/0256-307X/41/5/057403.
 65. Leonov, I. V. (2024). Electronic structure and magnetic correlations in trilayer nickelate superconductor $\text{La}_4\text{Ni}_3\text{O}_{10}$ under pressure. Phys. Rev. B **109**: 235123. DOI: 10.1103/PhysRevB.109.235123.
 66. Tian, P.-F., Ma, H.-T., Ming, X., et al. (2024). Effective model and electron correlations in trilayer nickelate superconductor $\text{La}_4\text{Ni}_3\text{O}_{10}$. J. Phys: Condens. Matter **36**: 355602. DOI: 10.1088/1361-648X/ad512c.
 67. Wang, J.-X., Ouyang, Z., He, R.-Q., et al. (2024). Non-Fermi liquid and Hund correlation in $\text{La}_4\text{Ni}_3\text{O}_{10}$ under high pressure. Phys. Rev. B **109**: 165140. DOI: 10.1103/PhysRevB.109.165140.
 68. LaBollita, H., Kapeghian, J., Norman, M. R., et al. (2024). Electronic structure and magnetic tendencies of trilayer $\text{La}_4\text{Ni}_3\text{O}_{10}$ under pressure: Structural transition, molecular orbitals, and layer differentiation. Phys. Rev. B **109**: 195151. DOI: 10.1103/PhysRevB.109.195151.
 69. Zhang, Y., Lin, L. F., Moreo, A., et al. (2024). Prediction of $s^{\pm}$ -wave superconductivity enhanced by electronic doping in trilayer nickelates $\text{La}_4\text{Ni}_3\text{O}_{10}$ under pressure. Phys. Rev. Lett. **133**: 136001. DOI: 10.1103/PhysRevLett.133.136001.
 70. Yang, Q. G., Jiang, K. Y., Wang, D., et al. (2024). Effective model and s_x -wave superconductivity in trilayer nickelate $\text{La}_4\text{Ni}_3\text{O}_{10}$. Phys. Rev. B **109**: L220506. DOI: 10.1103/PhysRevB.109.L220506.
 71. Lu, C., Pan, Z., Yang, F., et al. (2024). Superconductivity in $\text{La}_4\text{Ni}_3\text{O}_{10}$ under pressure. arXiv:2402.06450. DOI: 10.48550/arXiv.2402.06450.
 72. Luo, Z., Chen, C.-Q., Wang, M., et al. (2024). Trilayer multi-orbital models of $\text{La}_4\text{Ni}_3\text{O}_{10}$. Phys. Rev. B **110**: 014503. DOI: 10.1103/PhysRevB.110.014503.
 73. Zhang, M., Sun, H., Liu, Y. B., et al. (2024). The $s^{\pm}$ -wave superconductivity in the pressurized $\text{La}_4\text{Ni}_3\text{O}_{10}$. arXiv:2402.07902. DOI: 10.48550/arXiv.2402.07902.
 74. Scalapino, D. J. (2012). A common thread: The pairing interaction for unconventional superconductors. Rev. Mod. Phys. **84**: 1383. DOI: 10.1103/RevModPhys.84.1383.
 75. Wang, Z., Zou, C., Lin, C., et al. (2023). Correlating the charge-transfer gap to the maximum transition temperature in $\text{Bi}_2\text{Sr}_2\text{Ca}_{n-1}\text{Cu}_n\text{O}_{2n+4\delta}$. Science **381**: 227-231. DOI: 10.1126/science.add3672.
 76. Li, H., Zhou, X., Nummy, T., et al. (2017). Fermiology and electron dynamics of trilayer nickelate $\text{La}_4\text{Ni}_3\text{O}_{10}$. Nat. Commun. **8**: 704. DOI: 10.1038/s41467-017-00777-0.
 77. Mayr, M., Alvarez, G., Sen, C., et al. (2005). Phase fluctuations in strongly coupled d -wave superconductors. Phys. Rev. Lett. **94**: 217001. DOI: 10.1103/PhysRevLett.94.217001.
 78. Dubi, Y., Meir, Y. and Avishai, Y. (2007). Nature of the superconductor-insulator transition in disordered superconductors. Nature **449**: 876-880. DOI: 10.1038/nature06180.
 79. Karmakar, M. (2020). Pauli limited d -wave superconductors: quantum breached pair phase and thermal transitions. J. Phys.: Condens. Matter **32**: 405604. DOI: 10.1088/1361648X/ab926a.
 80. Pasrija, K., Chakraborty, P. B., and Kumar, S. (2016). Effective Hamiltonian based Monte Carlo for the BCS to BEC crossover in the attractive Hubbard model. Phys. Rev. B **94**: 165150. DOI: 10.1103/PhysRevB.94.165150.
 81. Dong, J.-J., Huang, D., and Yang, Y.-F. (2021). Mutual information, quantum phase transition, and phase coherence in Kondo systems. Phys. Rev. B **104**: L081115. DOI: 10.1103/PhysRevB.104.L081115.
 82. Dong, J.-J. and Yang, Y.-F. (2022). Development of long-range phase coherence on the Kondo lattice. Phys. Rev. B **106**: L161114. DOI: 10.1103/PhysRevB.106.L161114.
 83. Mukherjee, A., Patel, N. D., Dong, S., et al. (2014). Testing the Monte Carlo-mean field approximation in the one-band Hubbard model. Phys. Rev. B **90**: 205113. DOI: 10.1103/PhysRevB.90.205113.
 84. Liang, S., Moreo, A., and Dagotto, E. (2013). Nematic state of pnictides stabilized by interplay between spin, orbital, and lattice degrees of freedom. Phys. Rev. Lett. **111**: 047004. DOI: 10.1103/PhysRevLett.111.047004.
 85. Qin, Q., Dong, J.-J., Sheng, Y., Huang, D., et al. (2023). Superconducting fluctuations and charge-4e plaquette state at strong coupling. Phys. Rev. B **108**: 054506. DOI: 10.1103/PhysRevB.108.054506.
 86. Han, Q., Li, T., and Wang, Z. D. (2010). Pseudogap and Fermi-arc evolution in the phase-fluctuation scenario. Phys. Rev. B **82**: 052503. DOI: 10.1103/PhysRevB.82.052503.
 87. Zhong, Y. W., Li, T., and Han, Q. (2011). Monte Carlo Study of thermal fluctuations and Fermi-arc formation in d -wave superconductors. Phys. Rev. B **84**: 024522. DOI: 10.1103/PhysRevB.84.024522.
 88. Singh, D. K., Kadge, S., Bang, Y., et al. (2022). Fermi arcs and pseudogap phase in a minimal microscopic model of d -wave superconductivity. Phys. Rev. B **105**: 054501. DOI: 10.1103/PhysRevB.105.054501.
 89. Coleman, P. (2015). Introduction to many-body physics. (Cambridge University

- Press).
90. Berezinskii, V. L. (1972). Destruction of long-range order in one-dimensional and two-dimensional systems possessing a continuous symmetry group. II. Quantum Systems. *Sov. Phys. JETP* **34**: 610.
 91. Kosterlitz, J. M. and Thouless, D. J. (1973). Ordering, metastability and phase transitions in two-dimensional systems, *J. Phys. C* **6**: 1181. DOI: 10.1088/0022-3719/6/7/010.
 92. Kosterlitz, J. M. (1974). The critical properties of the two-dimensional XY model. *J. Phys. C* **7**: 1046. DOI: 10.1088/0022-3719/7/6/005.
 93. Drouin-Touchette, V. The Kosterlitz-Thouless phase transition: An introduction for the intrepid student. arXiv:2207.13748. DOI:10.48550/arXiv.2207.13748.
 94. Cover, T. M. and Thomas, J. A. (2006). *Elements of information theory*. (Wiley Series in Telecommunications and Signal Processing).
 95. Kraskov, A., Stögbauer, H., and Grassberger, P. (2004). Estimating mutual information. *Phys. Rev. E* **69**: 066138. DOI: 10.1103/PhysRevE.69.066138.
 96. Darbellay, G. A., and Vajda, I. (1999). Estimation of the information by an adaptive partitioning of the observation space. *IEEE Trans. Inf. Theory* **45**: 1315–1321. DOI: 10.1109/18.761290.
 97. Khan, S., Bandyopadhyay, S., Ganguly, A. R., et al. (2007). Relative performance of mutual information estimation methods for quantifying the dependence among short and noisy data. *Phys. Rev. E* **76**: 026209. DOI: 10.1103/PhysRevE.76.026209.
 98. Belghazi, M. I., Baratin, A., Rajeshwar, S., et al. (2018). Mutual information neural estimation. Dy, J. and Krause, A. (eds). *Proceedings of the 35th international conference on machine learning*, pp: 531.
 99. Poole, B., Ozair, S., Van Den Oord, A., et al. (2019). On variational bounds of mutual information. Chaudhuri, K. and Salakhutdinov, R. (eds). *Proceedings of the 36th international conference on machine learning*, pp: 5171.
 100. Le Tacon, M., Ghiringhelli, G., Chaloupka, J. et al. (2011). Intense paramagnon excitations in a large family of high-temperature superconductors. *Nat. Phys.* **7**: 725. DOI: 10.1038/nphys2041.
 101. Keimer, B., Kivelson, S., Norman, M. et al. (2015). From quantum matter to high-temperature superconductivity in copper oxides. *Nature* **518**: 179–186. DOI: 10.1038/nature14165.
 102. Emery, V. and Kivelson, S. (1995). Importance of phase fluctuations in superconductors with small superfluid density. *Nature* **374**: 434–437. DOI: 10.1038/374434a0.
 103. Qin, Q. and Yang, Y.-F. (2024). Intrinsic constraint on T_c for unconventional superconductivity. arXiv:2402.07128. DOI: 10.48550/arXiv.2402.07128.
 104. Chen, X., Zhang, J., Thind, A. S., et al. (2024). Polymorphism in Ruddlesden-Popper $\text{La}_3\text{Ni}_2\text{O}_7$: Discovery of a hidden phase with distinctive layer stacking. *J. Am. Chem. Soc.* **146**: 3640–3645. DOI: 10.1021/jacs.3c14052.
 105. Puphal, P., Reiss, P., Enderlein, N., et al. (2023). Unconventional crystal structure of the high-pressure superconductor $\text{La}_3\text{Ni}_2\text{O}_7$. arXiv:2312.07341. DOI: 10.48550/arXiv.2312.07341.
 106. Wang, H., Chen, L., Rutherford, A., et al. (2024). Long-range structural order in a hidden phase of Ruddlesden-Popper bilayer nickelates $\text{La}_3\text{Ni}_2\text{O}_7$. *Inorg. Chem.* **63**: 5020–5026. DOI: 10.1021/acs.inorgchem.3c04474.
 107. Li, J., Ma, P., Zhang, H., et al. (2024). Pressure-driven right-triangle shape superconductivity in bilayer nickelate $\text{La}_3\text{Ni}_2\text{O}_7$. arXiv:2404.11369. DOI: 10.48550/arXiv.2404.11369.

FUNDING AND ACKNOWLEDGMENTS

This work was supported by the Strategic Priority Research Program of the Chinese Academy of Sciences (Grant No. XDB33010100), the National Natural Science Foundation of China (Grants No. 12474136 and No. 12304174), and the National Key Research and Development Program of China (Grant No. 2022YFA1402203). The funders had no role in study design, data collection and analysis, decision to publish or preparation of the manuscript.

AUTHOR CONTRIBUTIONS

Y.Y. conceived the idea and directed the research. Q.Q. and J.W. performed the calculations. Q.Q., J.W., and Y.Y. wrote the paper.

DECLARATION OF INTERESTS

The authors declare no competing interests.

DATA AND CODE AVAILABILITY

The authors state that the primary data supporting the findings of this study are presented within the paper. Additional data and the code used to generate these data are available from the corresponding author upon reasonable request.

SUPPLEMENTAL INFORMATION

It can be found online at <https://doi.org/10.59717/j.xinn-mater.2024.100102>

LEAD CONTACT WEBSITE

<https://english.iop.cas.cn/pe/?id=1732>